ON CHAPLYGIN FUNCTIONS

(O FUNKTSIIAKH CHAPLYGINA)

PMM Vol.23, No.3, 1959, pp. 574-575

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(Received 24 January 1959)

1. The equation for the determination of the stream function of steady plane-parallel potential motion of a gas, expressed in Chaplygin's variables θ and τ, is

$$\frac{\partial}{\partial \tau}\left\{\frac{2\tau}{(1-\tau)^{\beta}}\frac{\partial \psi}{\partial \tau}\right\}+\frac{1-(2\beta+1)\tau}{2\tau(1-\tau)^{\beta+1}}\frac{\partial^2 \psi}{\partial \theta^2}=0$$

This has the following particular solution:

$$\psi(\theta,\tau)=z_n(\tau)\sin(n\theta+\alpha)$$

in which $z_n(\tau)$ is given in terms of the integral $y_n(\tau)$ of the hypergeometric equation

$$\tau(1-\tau)\frac{d^2 y_n}{d\tau^2}+[(n+1)+(\beta-n-1)\tau]\frac{dy_n}{d\tau}+\frac{1}{2}\beta n(n+1)y_n=0 \qquad (1)$$

by the formula

$$z_n(\tau)=\tau^{\frac{1}{2}}y_n(\tau)$$

Equation (1) has a particular solution represented by the hypergeometric series

$$F(a_n, b_n, c_n; \tau) \qquad (2)$$

in which the parameters a_n, b_n, c_n are determined from the equations

$$a_n+b_n=n-\beta, \qquad a_n b_n=-\frac{1}{2}\beta n(n+1), \qquad c_n=n+1$$

In what follows we shall consider only this particular solution of equation (1); the second particular solution, independent of the first, can be calculated from the first by means of a quadrature.

In order to determine the parameters a_n and b_n separately it is necessary to solve the quadratic equation

$$\rho^2-(n-\beta)\rho-\frac{1}{2}\beta n(n+1)=0$$

the roots of which in fact give the parameters a_n and b_n.

In what follows we shall assume that the number β is equal to 5/2. Under this assumption the values of the parameters a_n and b_n are given by the formulas

$$a_n = \frac{1}{2}\left(n - \frac{5}{2} + \frac{1}{2}\sigma\right), \qquad b_n = \frac{1}{2}\left(n - \frac{5}{2} - \frac{1}{2}\sigma\right) \tag{3}$$

where σ has the following value:

$$\sigma = \sqrt{24n^2 + 25}$$

The aim of the present note is to find values of the index n, for which the number σ is an integral rational number. The interest in determining such integral numbers n, for which the number σ is also an integer, consists in the fact that, for these numbers n, Chaplygin functions $z_n(\tau)$ can be expressed in terms of elementary functions.

As it happens, the number σ is always odd: $\sigma = 2\sigma' + 1$; hence, formulas (3) can be rewritten thus:

$$a_n = \frac{1}{2}(n + \sigma' - 2), \qquad b_n = \frac{1}{2}(n - \sigma' - 3)$$

Since the number σ is larger than $4n$, the parameter b_n is always negative, and if the numbers n and σ' are of different parity then the number b_n is an integral negative number; accordingly, the hypergeometric series (2) becomes a polynomial.

If, however, the numbers n and σ' are of the same parity, then the number a_n is an integral positive number; the number b_n is equal to half of an odd number; by virtue of this, the hypergeometric series (2) reduces to a rational function of the variable τ. We notice that in this case the second solution of equation (1) is also expressed [1] by elementary functions of the variable τ.

2. Now our problem consists in finding for what integral numbers n the number σ is also an integer. For the solution of this problem we have to find integral solutions of the indeterminate equation:

$$\sigma^2 - 24n^2 = 25 \tag{4}$$

The solution of this equation can be obtained using methods described by Dirichlet in his book on the theory of numbers [2].

In the solution of equation (4) it is necessary to distinguish two cases: the first case when the numbers n and σ are mutually prime; the second case when the numbers n and σ have a greatest common factor different from unity. Such a factor can be only the number 5; putting $n = 5n'$, $\sigma = 5\sigma'$, we obtain the following equation for determining n' and σ':

$$\sigma'^2 - 24n'^2 = 1 \tag{5}$$

Applying the methods described in the book mentioned above, we obtain in the first case the following solutions of equation (4):

$$\sigma_k = 7\,t_k + 24\,u_k, \qquad n_k = -\,(t_k + 7\,u_k) \tag{6}$$

and

$$\sigma_k = 7\,t_k - 24\,u_k, \qquad n_k = t_k - 7\,u_k \tag{7}$$

In each of these pairs of formulas the number k takes all positive integral values from 0 to ∞. As regards t_k and u_k, these are arbitrary pairs of solutions of Pell's equation

$$t^2 - 24\,u^2 = 1$$

The smallest pair of numbers satisfying this equation is $T = 5$, $U = 1$. All other pairs t_k, u_k which are solutions of Pell's equation are determined by means of the remarkable formula

$$t_k + u_k\sqrt{24} = (5 + 1.\sqrt{24})^k \tag{8}$$

where $k = 0, 1, 2, 3, \ldots$ The values of the numbers t_k and u_k for certain values of the index k are given in Section 3.

If the number σ or n, found from formulas (6) or (7), turns out to be negative, then we can take the positive number which is equal to this number in absolute magnitude. On the basis of formulas (6) and (7) we construct the following Table of values of the numbers σ, n, a_n, b_n, c_n:

$k = 0$	$n = 1$	$\sigma = 7$	$a_1 = 1$	$b_1 = -\frac{5}{2}$	$c_1 = 2$
$k_1 = 1$	$n = 2$	$\sigma = 11$	$a_2 = \frac{5}{2}$	$b_2 = -3$	$c_2 = 3$
	$n = 12$	$\sigma = 59$	$a_{12} = \frac{39}{2}$	$b_{12} = -10$	$c_{12} = 13$
$k_2 = 2$	$n = 21$	$\sigma = 103$	$a_{21} = 35$	$b_{21} = -\frac{33}{2}$	$c_{21} = 22$
	$n = 119$	$\sigma = 583$	$a_{119} = 204$	$b_{119} = -\frac{175}{2}$	$c_{119} = 120$
$k = 3$	$n = 208$	$\sigma = 1019$	$a_{208} = \frac{715}{2}$	$b_{208} = -152$	$c_{208} = 209$
	$n = 1178$	$\sigma = 5771$	$a_{1178} = \frac{4061}{2}$	$b_{1178} = -855$	$c_{1178} = 1179$

3. Let us turn now to the solution of equation (5). On the basis of formula (8) we obtain the following values for the numbers σ' and n', equal respectively to the numbers t and u:

$k = 0$	$k = 1$	$k = 2$	$k = 3.$	..
$t = 1$	$t = 5$	$t = 49$	$t = 485.$	..
$u = 0$	$u = 1$	$u = 10$	$u = 99.$	..

From these numbers we can construct the following final Table:

$k=0$	$n=0$	$\sigma=5$	$a_0=0$	$b_0=-\frac{5}{2}$	$c_0=1$
$k=1$	$n=5$	$\sigma=25$	$a_5=\frac{15}{2}$	$b_5=-5$	$c_5=6$
$k=2$	$n=50$	$\sigma=245$	$a_{50}=85$	$b_{50}=-\frac{75}{2}$	$c_{50}=51$
$k=3$	$n=495$	$\sigma=2425$	$a_{495}=\frac{2705}{2}$	$b_{495}=-360$	$c_{495}=496$

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2. Lejeune-Dirichlet, P.G., *Lektsii po teorii chisel (Lectures on the Theory of Numbers)*, Chap. IV, ONTI, Moscow, 1936.